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Generalized M-vector models for hedging interest rate risk

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Abstract

This paper generalizes the M-square and M-vector models [Fong and Fabozzi, Appendix E: Derivation of Risk Immunization Measures, in: Fixed Income Portfolio Management, Dow Jones-Irwin, Homewood, IL, pp. 291–294, 1985; Nawalkha and Chambers, Journal of Portfolio Management, Winter (1997) 92] by using a Taylor series expansion of the bond return function with respect to specific functions of the cash flow maturities. The classic M-vector computes the weighted averages of the distance between the maturity of each cash flow and the planning horizon, raised to integer powers (e.g., $(t-H)^1, (t-H)^2, (t-H)^3, \ldots$). Implementation of the new approach involves computing the weighted averages of the distance between some function of the maturity of each cash flow and that of the planning horizon, raised to integer powers (e.g., $(g(t)-g(H))^1, (g(t)-g(H))^2, (g(t)-g(H))^3, \ldots$). Adopting this approach, this paper explores six different generalized M-vector models corresponding to six different polynomial functions, over five different planning horizons from one year to five years. It is shown that generalized M-vector models corresponding to polynomial functions of lower power provide significantly enhanced protection from interest rate risk over short planning horizons.

JEL classification: E43; G11
Keywords: Immunization; Duration; Interest rate; Risk management; Fixed income

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1. Introduction

Interest rate immunization against arbitrary non-parallel term structure shifts can be achieved using two classes of duration models – *key-rate duration* models and *duration-vector* models. [1] Key-rate duration models have been developed by Ho (1992), Jarrow and Turnbull (1994), and Dattatreya and Fabozzi (1995). These models develop multiple duration measures by modeling the term structure of interest as a sum of approximately linear segments. Key-rate duration models allow any number of risk measures, and therefore, in theory, can hedge interest rate risk to any desired level of precision. However, the number of risk measures to be used and the corresponding division of the term structure into different key rates remain quite arbitrary under these models. For example, Ho (1992) proposes as many as *eleven* key rate durations to effectively hedge against interest rate risk.

Duration vector models use a vector of higher-order duration measures to immunize against changes in the shape parameters of the term structure of interest rates. Vector models have been developed by Cooper (1977), Chambers (1981), Granito (1984), Bierwag et al. (1987), Chambers et al. (1988), Prisman and Shores (1988), Prisman and Tian (1994), Barrett et al. (1995), and Barber and Copper (1996) using particular functional forms for the term structure or its shifts. Unlike the key-rate duration models, the duration vector models provide a high level of immunization performance using only three to five risk measures. More generalized forms of duration vector models are given by Fong and Fabozzi (1985) and Nawalkha and Chambers (1997). [2] Unlike the traditional duration vector models, the M-square model of Fong and Fabozzi (1985) and the M-vector model of Nawalkha and Chambers (1997) do not restrict the term structure shifts to be of a particular functional form. These models are based upon a Taylor series expansion of the bond return function with respect to the cash flow maturities around a given planning horizon and lead to duration measures that are linear in $t, t^2, t^3, \ldots$ where t is the maturity of the cash flow.

However, since interest rate shifts are generally larger at the shorter end of the maturity spectrum, its possible that an alternative set of duration measures that are linear in $g(t), g(t)^2, g(t)^3, \ldots$, such that $g(t)^m$ $(m = 1, 2, 3, \ldots)$ puts relatively more weight at the shorter end of the maturity spectrum may give enhanced immunization protection.

Consistent with this intuition, this paper derives generalized M-vector models using a Taylor series expansion of the bond return function with respect to specific functions of the cash flow maturities. Using a change of variable for the Taylor series expansion, the mth-order element of the generalized M-vector model is derived as a weighted average of $(g(t) - g(H))^m$ where t is the maturity of the cash flow, and H is the immunization horizon.

[1] See Nawalkha and Chambers (1999).

[2] These models are derived using a Taylor series approach, which allows the risk measures to be set to zero. In contrast, the M-square model of Fong and Vasicek (1984) and the M-absolute model of Nawalkha and Chambers (1996) are derived to minimize the risk measures subject to portfolio constraints.

In order to empirically investigate the generalized M-vector models, we perform empirical tests on a class of polynomial functions given as $g(t) = t^{\alpha}$. Different values of the parameter α result in different generalized M-vector models. The value of $\alpha = 1$ corresponds to the case of the classic M-vector where the distance between the maturity of each cash flow and planning horizon is raised to integer powers (e.g., $(t-H)^1, (t-H)^2, (t-H)^3, \ldots$). Other values of α are equivalent to cases where the distance between some polynomial function of the maturity of each cash flow and that of the planning horizon, is raised to integer powers (e.g., $\alpha = 0.5$ produces $(t^{0.5} - H^{0.5})^1, (t^{0.5} - H^{0.5})^2, (t^{0.5} - H^{0.5})^3, \ldots$).

We test six different generalized M-vector models corresponding to six different values of α given as 0.25, 0.50, 0.75, 1, 1.25, and 1.5 over five different horizons of one year, two years, three years, four years, and five years. Using McCulloch term structure data over 1950–1990 to form bond portfolios that are rebalanced annually, we find that generalized higher-order M-vector strategies with lower (higher) α significantly outperform (underperform) the classic M-vector strategy over short planning horizons, but these strategies are insignificantly different from the classic M-vector strategy over long horizons.

The rest of the paper is organized as follows: Section 2 derives the generalized M-vector models; Section 3 presents the data, methodology and results of the empirical tests; Section 4 concludes the paper.

2. The generalized M-vector models

Consider a bond portfolio at time $t = 0$ with C_t as the payment on the portfolio at time t $(t = t(1), t(2), \ldots, t(N))$. Let the continuously compounded instantaneous forward rate function at time $t = 0$ be given by $i(t)$. Now allow an instantaneous shift in forward rates from $i(t)$ to $i'(t)$ such that $i'(t) = i(t) + \Delta i(t)$. The return on the bond portfolio between $t = 0$ and $t = H$ can be given as

$$R(H) = \frac{T_H - P_0}{P_0} \tag{1}$$

where

$$P_0 = \sum_{t=t(1)}^{t(N)} C_t W_t$$

is the value of the bond portfolio at $t = 0$;

$$W_t = \exp\left[-\int_0^t i(\tau)\,\mathrm{d}\tau\right]$$

is the discount function;

$$T_H = \sum_{t=t(1)}^{t(N)} C_t \cdot \exp\left[\int_t^H i'(\tau)\,\mathrm{d}\tau\right]$$

is the terminal value of the portfolio.

Substituting the definitions of P_0 and T_H into Eq. (1) gives

$$R(H) = \left[\exp\left[\int_0^H i(\tau)\,\mathrm{d}\tau\right]\left[\sum_{t=t(1)}^{t(N)} C_t \cdot W_t \cdot f(t)\right] - P_0\right] \Big/ P_0 \tag{2}$$

where $f(t) = \exp\left[\int_t^H \Delta i(\tau)\,\mathrm{d}\tau\right]$.

Using a change of variable, let the forward rate function $i(t)$ be represented by a chain function given as

$$i(t) = h(g(t)) \tag{3}$$

where $g(t)$ is a continuously differentiable function of t. Further assume that $g(t)$ is monotonic and the inverse function of $g(t)$ exists and is given as

$$t = g^{-1}(g) = k(g). \tag{4}$$

The instantaneous change in the forward rate function can be given as

$$\Delta i(t) = \Delta h(g(t)). \tag{5}$$

Using Eqs. (4) and (5), we have

$$f(t) = \exp\left[\int_t^H \Delta i(\tau)\,\mathrm{d}\tau\right] = \exp\left[\int_{g(t)}^{g(H)} p(\gamma)\,\mathrm{d}\gamma\right] = r(g(t)) \tag{6}$$

where

$$p(g(t)) = \Delta h(g(t)) \cdot \frac{\partial(k(g))}{\partial g}. \tag{7}$$

Doing a Taylor series expansion of $r(g(t))$ around $g(H)$, $f(t)$ can be represented as

$$\begin{aligned} f(t) &= r(g(t)) \\ &= 1 - [g(t) - g(H)] \cdot p(g(H)) \\ &\quad - \frac{1}{2}[g(t) - g(H)]^2 \cdot \left[\frac{\partial(p(g))}{\partial g} - [p(g)]^2\right]_{g=g(H)} \\ &\quad - \frac{1}{3!}[g(t) - g(H)]^3 \cdot \left[[p(g)]^3 - 3 \cdot p(g)\frac{\partial(p(g))}{\partial g} + \frac{\partial^2(p(g))}{\partial g^2}\right]_{g=g(H)} \\ &\quad + \cdots + \\ &\quad - \frac{1}{Q!}[g(t) - g(H)]^Q \cdot \left[(-1)^{Q+1}[p(g)]^Q + \cdots + \frac{\partial^{Q-1}(p(g))}{\partial g^{Q-1}}\right]_{g=g(H)} \\ &\quad + \cdots + \text{remainder}. \end{aligned} \tag{8}$$

For a reasonably large number Q, the first $Q+1$ terms of the above equation may approximate the value of $r(g(t))$ well. Eq. (8) can be written in a simplified form as

$$f(t) = r(g(t)) = 1 + \sum_{m=1}^{Q} [g(t) - g(H)]^m \cdot Z_m + \varepsilon(t) \tag{9}$$

where

$$\begin{aligned}
Z_1 &= -p(g(H)),\\
Z_2 &= -\frac{1}{2} \cdot \left[\frac{\partial(p(g))}{\partial g} - [p(g)]^2 \right]_{g=g(H)},\\
Z_3 &= -\frac{1}{3!} \cdot \left[[p(g)]^3 - 3 \cdot p(g) \frac{\partial(p(g))}{\partial g} + \frac{\partial^2(p(g))}{\partial g^2} \right]_{g=g(H)},\\
&\ \vdots\\
Z_Q &= -\frac{1}{Q!} \cdot \left[(-1)^{Q+1} [p(g)]^Q + \cdots + \frac{\partial^{Q-1}(p(g))}{\partial g^{Q-1}} \right]_{g=g(H)}.
\end{aligned}$$

The expression $\varepsilon(t)$ is the error term due to higher-order Taylor series terms.

Eq. (7) shows that the value of $p(g)$ depends on the change in the forward rate function. In particular, if the forward rate function does not change, $p(g(H))$ equals zero and therefore $Z_m = 0$ for all $m = 1, 2, \ldots, Q$. In this case, $f(t) = r(g(t)) = 1$, and the return on the portfolio is riskless. Substituting $f(t) = 1$ into Eq. (2), the riskless return between time 0 and H is given as

$$R_F(H) = \exp\left[\int_0^H i(\tau)\, \mathrm{d}\tau \right] - 1. \tag{10}$$

If forward rates do change, the bond portfolio return will be different from the riskless return. The bond portfolio return can be obtained by substituting Eq. (9) into Eq. (2), which gives

$$R(H) = R_F(H) + [1 + R_F(H)] \cdot \sum_{m=1}^{Q} Z_m M^m + \varepsilon_R \tag{11}$$

where ε_R is the error term due to higher-order Taylor series terms, $R_F(H)$ is the riskless return defined in Eq. (10), and M^m is the mth measure of the generalized M-vector corresponding to a given function $g(t)$ for all $m = 1, 2, \ldots, Q$. The mth measure is of the following form:

$$M^m = \left[\sum_{t=t(1)}^{t(N)} C_t \cdot W_t \cdot [g(t) - g(H)]^m / P_0 \right]. \tag{12}$$

The portfolio return can be rewritten as

$$R(H) = R_F(H) + \mathbf{M} \cdot \mathbf{Y}' + \varepsilon_R \tag{13}$$

where

$$\mathbf{M} = |M^1, M^2, \ldots, M^Q|,$$
$$\mathbf{Y} = |Y_1, Y_2, \ldots, Y_Q| \quad \text{and}$$
$$Y_m = (1 + R_F(H)) \cdot Z_m \quad \text{for all } m = 1, 2, \ldots, Q.$$

Eq. (13) allows us to separate the total return of a default free bond portfolio into two parts. The first part is the riskless return $R_F(H)$. The second part is the unexpected return due to the instantaneous change in the forward rates. The unexpected return can be considered as the composite result of two effects: the shift vector $\mathbf{Y}$ which measures the impact of the changes in forward rates at the planning horizon H, and the generalized M-vector $\mathbf{M}$ which is determined by the maturity characteristics of the bond portfolio. The separation of the forward rate changes (captured by the shift vector) and the maturity characteristics (captured by the generalized M-vector) makes it possible to immunize a bond portfolio from unexpected changes in the forward rates. This is achieved by equating the generalized M-vector $\mathbf{M}$ to a vector of zeros.

Eq. (13) is very general and allows flexibility in choosing various functional forms of $g(t)$. In this paper, we test a class of polynomial functions given by $g(t) = t^\alpha$ for six different values of α equal to 0.25, 0.5, 0.75, 1, 1.25, and 1.5, respectively. We choose the polynomial function for two reasons. First, because of its simplicity, and second, for a range of positive values of alpha (i.e., $0 < \alpha < 1$), the polynomial function has some desirable properties given as follows. When $Q = 1$, the specific values of α between 0 and 1 imply that duration of zero coupon bonds is increasing, though at a decreasing rate, or $\mathrm{d}g(t)/\mathrm{d}t > 0$ and $\mathrm{d}^2g(t)/\mathrm{d}t^2 < 0$. This property is also satisfied by all known term structure models with mean reversion that allow short rates to move more than long rates (e.g., Vasicek, 1977; Cox et al., 1985). On the other hand, values of α greater than 1 imply that duration of zero coupon bonds is increasing at an *increasing* rate (i.e., $\mathrm{d}^2g(t)/\mathrm{d}t^2 > 0$). In the context of term structure models, this can happen only under mean *aversion* implying short rates move *less* than long rates.

Though the generalized M-vector models do not require the restrictive assumptions of the term structure models (since Q may be greater than 1, allowing more factors and moments of forward rate changes), it is still important to note these relations in order to explore potential functional forms for $g(t)$, which remain consistent with the basic term structure literature under the special case.

An intuitive reason for preferring values of α between 0 and 1, to values of α greater than 1 is as follows. Duration vector models derived from Taylor series expansion lead to duration measures that are linear in $t, t^2, t^3, \ldots$ However, since interest rate shifts are generally larger at the shorter end of the maturity spectrum, its possible that an alternative set of duration measures that are linear in $g(t)^m$ (for $m = 1, 2, 3, \ldots$), such that $g(t)^m$ puts relatively more weight at the shorter end of the maturity spectrum may give enhanced immunization protection for increasing values of Q. Mathematically, this condition can be expressed as $g(s)^m/g(t)^m < s^m/t^m$, for $m = 1, 2, 3, \ldots$, where $s > t$. It can be verified that for the polynomial function $g(t) = t^\alpha$, this condition is satisfied only when $\alpha < 1$.

Finally, the reason we do not consider negative values of α is because they imply that the corresponding first-order duration measure is decreasing with bond maturity, and hence, when $Q = 1$ the zero coupon bond volatility must decrease with increasing maturity, which is obviously not true.

3. Empirical tests

In this section, we perform a set of empirical tests over the observation period 1950 through 1990 to examine the immunization performance of alternative M-vector models corresponding to $g(t) = t^{\alpha}$ with six different values of α: 0.25, 0.5, 0.75, 1, 1.25 and 1.5.

The case of $\alpha = 1$ is equivalent to the case of the classic M-vector where the distance between the maturity of each cash flow and the planning horizon is raised to integer powers (e.g., $(t - H)^1, (t - H)^2, (t - H)^3, \ldots$). Nawalkha and Chambers (1997) provide empirical evidence supporting this model using a planning horizon of four years. Our empirical tests not only confirm their findings, but also extend their model by considering other values of α equivalent to cases where the distance between some polynomial function of the maturity of each cash flow and that of the planning horizon, is raised to integer powers (e.g., $\alpha = 0.5$ produces $(t^{0.5} - H^{0.5})^1$, $(t^{0.5} - H^{0.5})^2, (t^{0.5} - H^{0.5})^3, \ldots$). Further, the generalized M-vector models are tested against the classic M-vector using an increasing number of immunization constraints (i.e., Q ranging from 1 to 5) [3] over five different planning horizons of one year, two years, three years, four years, and five years.

3.1. Data

The empirical tests are based upon the McCulloch US Treasury term structure data. The zero-coupon yields of various maturities for period 1947 through 1991 are derived by McCulloch and Kwon (1993) using a broad spectrum of government bond prices. [4] This data set has been used before for empirical studies on bond immunization (see Elton et al., 1990; Nawalkha and Chambers, 1997).

The immunization tests use the zero-coupon yields with maturities ranging from one year through seven years. The yields are recorded on December 31 of each year from 1950 to 1990. Thirty-one coupon bonds with annual coupon payments are constructed using the zero-coupon yields. The coupon bonds have seven maturities $(1, 2, 3, \ldots, 7$ years), and there are five different coupon rates (6%, 8%, 10%, 12% and 14%) for each maturity. [5]

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[3] Higher values of Q would result in a high sensitivity of the results to round-off errors.

[4] For details of the term structure estimation, see McCulloch (1971, 1975) and Kwon (1992).

[5] Since coupons are paid annually, all the one-year maturity coupon bonds collapse to a single one-year zero-coupon bond regardless of the coupon rate.

3.2. Methodology

For each planning horizon of length H, the sample period beginning December 31, 1950 through December 31, 1990 is divided into $41 - H$ number of H-year planning periods. For example, for $H = 2$, the sample period is divided into 39 overlapping two-year planning periods, given as $1950–1952, 1951–1953, \ldots, 1988–1990$. Similarly, for $H = 5$, the sample period is divided into 36 overlapping five-year planning periods, given as $1950–1955, 1951–1956, \ldots, 1985–1990$. [6] Given a specific planning horizon, a separate portfolio with an initial value of \$1 is formed at the beginning of each planning period (i.e., on December 31 of each year) corresponding to the six different functional forms of $g(t)$ (i.e., the six values of α: 0.25, 0.5, 0.75, 1, 1.25 and 1.5) and the five generalized M-vector lengths (i.e., Q ranging from 1 to 5). Since there are an infinite number of portfolios that would satisfy the constraints of each immunization strategy, we select a unique bond portfolio corresponding to each strategy by optimizing the following quadratic function:

$$\text{Min}\left[\sum_{i=1}^{J} p_i^2\right] \tag{14}$$

subject to:

$$\begin{aligned} &\sum_{i=1}^{J} p_i M_i^m = 0 \quad \text{for all } m = 1, 2, \ldots, Q, \quad \text{and} \\ &\sum_{i=1}^{J} p_i = 1 \end{aligned} \tag{15}$$

where $J = 31$ is the number of bonds in the portfolio and p_i is the proportion of the total wealth invested in the ith bond.

The objective function in (14) is used for achieving diversification across all bonds. The portfolios are rebalanced on December 31 of each year when annual coupons are paid. At the end of the H-year planning period, the terminal values of each of the portfolios corresponding to different functional forms of $g(t)$ and different lengths of the generalized M-vector model are compared with the target riskless return earned from holding a H-year zero-coupon bond issued at the beginning of the planning period. The deviations of the portfolio actual terminal values from the target terminal values are used to evaluate the effectiveness of immunization strategies under the different generalized M-vector models.

In order to evaluate the statistical significance of the improvement (or deterioration) of immunization performance, we carry out two non-parametric tests for related samples on the series of absolute deviations of actual values from target

[6] Since the portfolios are rebalanced at the end of each year, $H = 1$ gives 40 *non-overlapping* 1-year periods given as $1950–1951, 1951–1952, \ldots, 1989–1990$.

values from each generalized M-vector strategy with respect to the corresponding deviations from the classic M-vector strategy.

The first non-parametric test that we apply to our data is the *sign test*, which has been previously used in the immunization literature by Bierwag et al. (1993), and Fooladi and Roberts (1992). The sign test determines whether the percentage of planning periods in which each strategy outperforms the classic M-vector strategy is significantly different from a randomly expected outcome of 50%.

To gain more confidence in our results, we also apply the *Wilcoxon signed-rank test*, which is a non-parametric alternative to the paired samples t-test. This test complements the results of the sign test by taking into account, not only the sign, but also a comparison of the magnitudes of the absolute deviations from each strategy with the magnitudes of the corresponding deviations from the classic M-vector strategy. This test ranks the absolute differences between the absolute deviations from both strategies, and compares the sum of the ranks corresponding to the cases where the generalized M-vector strategy deviations are lower than the classic M-vector strategy deviations (negative ranks) to the sum of ranks corresponding to the opposite cases (positive ranks). The null hypothesis is that if both strategies perform similarly, the sum of ranks from both strategies must be approximately equal. [7]

3.3. Results

The results from the immunization tests are divided into two subsections. Section 3.3.1 reports a comparison of the deviations of actual portfolio values from target portfolio values from the generalized M-vector strategies against the deviations from the classic M-vector strategy for different vector lengths ranging from $Q = 1$ to $Q = 5$ over a planning horizon of three years. Section 3.3.2 uses the two non-parametric tests to assess the statistical significance of the improvement (or deterioration) in the immunization performance of generalized M-vector models over the classic M-vector model. All tests are repeated over different lengths of planning horizons ($H = 1$, 2, 3, 4 and 5 years). Repeating these tests over different planning horizons provides evidence of improvement in immunization performance, not only in a statistical sense, but also in an economic sense for the case when planning horizons are short. Finally, we provide an empirical insight based upon portfolio design that explains why the generalized M-vector models corresponding to polynomial functional forms with low alphas provide enhanced immunization performance over short planning horizons.

3.3.1. Immunization performance of the generalized M-vector models: A first look

Table 1 provides a first look at the immunization performance of the generalized M-vector models over the classic M-vector model for different vector lengths ranging

[7] For both the sign test and the Wilcoxon signed-rank test we ignore ties between deviations because none was found with a precision of ten digits.

from $Q = 1$ to $Q = 5$ when the length of the planning horizons is set to three years. [8] This table gives the sums of absolute deviations and the sums of negative deviations [9] of portfolio terminal values from the corresponding target values for the six functional forms of $g(t)$ as a percentage of the same sums for the classic M-vector model (i.e., with $g(t) = t$). The figures correspond to the sub-period 1950–1970 (Panel A), the higher volatility sub-period 1970–1990 (Panel B), and the whole period 1950–1990 (Panel C).

Inspection of the results for $Q = 1$ reveals that the three models $g(t) = t^{0.5}$, $g(t) = t^{0.75}$ and $g(t) = t$ have similar performance over the whole period, while the remaining three models $g(t) = t^{0.25}$, $g(t) = t^{1.25}$ and $g(t) = t^{1.5}$ perform worse over the whole period. When we examine the performance of the models in the separate sub-periods (Panel A and Panel B, respectively), we find that over the period 1950–1970, the models with α less than 1 outperform the models with α greater or equal to 1, but over period 1970–1990, the model $g(t) = t$ outperforms all other models. Hence, in this case, the findings are consistent with the previous research, that the immunization performance of the Macaulay duration model (i.e., using $g(t) = t$) is about as effective as that of other alternative single factor duration models (see, Bierwag et al., 1981; Brennan and Schwartz, 1983; Nelson and Schaefer, 1983; Gultekin and Rogalski, 1984).

When higher-order generalized M-vectors (corresponding to $Q = 2$, $Q = 3$, $Q = 4$, and $Q = 5$) are used, Table 1 shows that the lower α models generally outperform the classic model (i.e., with $\alpha = 1$) as well as other higher α models based upon the sum of absolute deviations. This holds for the both sub-periods and the whole sample period. This improvement in immunization performance seems very significant when $Q = 5$. The evidence considering the sum of negative deviations is more mixed, which reveals that the distribution of deviations is not symmetrical in this case. Over the whole period, the immunization performance does improve for lower alpha strategies with vector lengths of $Q = 2$, $Q = 4$, and $Q = 5$, but the classic M-vector performs as well as other strategies for $Q = 3$.

Though a pattern exists between the size of α and the immunization performance, this may not be *sufficient* for assessing whether the improvement (deterioration) in performance is statistically and/or economically significant. Also, the results shown are specific to the three-year planning horizon and may or may not hold at shorter or longer horizons. These important issues are addressed in the following section.

[8] Section 3.3.2 provides summary results and statistical tests for other planning horizons including, 1, 2, 3, 4, and 5 years.

[9] Absolute deviations can assess the immunization performance regardless of whether the immunizing portfolio represents a long position (managing assets to fund a known liability) or a short position (to hedge a liability portfolio against a known future cash inflow). Negative deviations are not valid for assessing the immunization performance of a portfolio that represents an overall *short* position (e.g., certain financial institutions with liability portfolios and hedge funds).

Table 1
Deviations of actual values from target values for generalized M-vector strategies as a percentage of the deviations for the classic M-vector strategy over three-year horizons

	$g(t)=t^{0.25}$	$g(t)=t^{0.5}$	$g(t)=t^{0.75}$	$g(t)=t$	$g(t)=t^{1.25}$	$g(t)=t^{1.5}$
Panel A: Observation period 1950–1970						
$Q=1$						
Absolute deviations	68.44	59.56	76.45	100.00	120.95	140.11
Negative deviations	46.45	35.29	62.07	100.00	135.73	170.07
$Q=2$						
Absolute deviations	83.64	79.73	82.39	100.00	134.82	182.42
Negative deviations	137.18	113.88	95.86	100.00	133.28	191.31
$Q=3$						
Absolute deviations	77.38	75.53	80.28	100.00	132.48	219.89
Negative deviations	94.48	95.69	96.03	100.00	103.45	152.93
$Q=4$						
Absolute deviations	52.27	71.20	91.47	100.00	106.67	176.27
Negative deviations	48.66	67.05	88.51	100.00	100.77	127.97
$Q=5$						
Absolute deviations	19.10	33.71	55.06	100.00	135.96	202.25
Negative deviations	12.50	26.79	50.00	100.00	142.86	205.36
Panel B: Observation period 1970–1990						
$Q=1$						
Absolute deviations	155.81	126.35	105.12	100.00	104.24	126.29
Negative deviations	175.89	138.62	110.31	100.00	100.26	120.69
$Q=2$						
Absolute deviations	57.45	55.79	72.04	100.00	123.26	149.55
Negative deviations	66.30	61.91	75.34	100.00	118.77	140.30
$Q=3$						
Absolute deviations	85.78	85.78	92.71	100.00	121.62	167.68
Negative deviations	98.10	95.97	99.35	100.00	112.87	149.29
$Q=4$						
Absolute deviations	70.77	79.48	90.26	100.00	111.35	124.44
Negative deviations	79.97	87.31	95.04	100.00	103.37	103.70
$Q=5$						
Absolute deviations	35.16	49.61	70.33	100.00	134.54	172.68
Negative deviations	24.52	41.60	66.12	100.00	136.91	171.63
Panel C: Summary for period 1950–1990						
$Q=1$						
Absolute deviations	118.57	100.16	94.26	100.00	110.46	131.08
Negative deviations	130.02	105.38	95.71	100.00	110.18	132.61
$Q=2$						
Absolute deviations	65.96	63.77	74.68	100.00	128.19	165.24
Negative deviations	78.33	71.92	78.53	100.00	124.09	158.03
$Q=3$						
Absolute deviations	85.22	83.04	88.88	100.00	122.12	174.36
Negative deviations	107.43	99.88	98.50	100.00	110.84	152.94
$Q=4$						
Absolute deviations	72.52	83.38	93.90	100.00	110.22	128.02
Negative deviations	81.23	91.48	99.23	100.00	98.91	98.09

(*continued on next page*)

Table 1 (*continued*)

	$g(t)=t^{0.25}$	$g(t)=t^{0.5}$	$g(t)=t^{0.75}$	$g(t)=t$	$g(t)=t^{1.25}$	$g(t)=t^{1.5}$
$Q=5$						
Absolute deviations	30.97	46.10	68.67	100.00	132.77	166.27
Negative deviations	21.37	38.83	65.03	100.00	134.35	160.94

This table reports the sum of absolute deviations and the sum of negative deviations of portfolio actual values from target values for each generalized M-vector strategy as a percentage of the corresponding sums for the classic M-vector strategy (i.e., with $g(t)=t$). The length of the planning horizon is three years. Each of the two sub-periods is comprised of 18 overlapped planning periods of three years (and thus, 18 separate portfolios with an initial value of \$1) and the whole period 1950–1990 is comprised of 38 overlapped planning periods of the same length. Percentages lower than 100% are marked with gray cells.

3.3.2. Non-parametric tests of immunization performance

This section uses two non-parametric tests to assess the statistical significance of the improvement (or deterioration) in the immunization performance of generalized M-vector strategies over the classic M-vector strategy. All tests are repeated over different lengths of planning horizons of 1, 2, 3, 4, and 5 years. This extends our basic results in Section 3.3.1, which were specific to a three-year planning horizon. We also find that repeating these tests over different planning horizons provides some evidence of improvement in immunization performance, not only in a statistical sense, but also in an economic sense.

The results of two non-parametric tests are presented in Tables 2–6 corresponding to planning horizons of increasing lengths of 1, 2, 3, 4, and 5 years. Each table reports the non-parametric tests of improvement (or deterioration) in the immunization performance of 25 different strategies (5×5 corresponding to different values of $\alpha = 0.25$, 0.5, 0.75, 1.25, and 1.5, and different values of $Q = 1$, 2, 3, 4, and 5) over the corresponding five classic M-vector strategies (i.e., $\alpha = 1$, and different values of $Q = 1$, 2, 3, 4, and 5).

Each table is subdivided into five panels (Panel A to Panel E) corresponding to increasing lengths of the generalized M-vector model (Q). Each panel reports the sum of absolute deviations of actual portfolio values from target values over the whole sample period. The sum of absolute deviations is also expressed as a percentage of the corresponding sum for the classic M-vector model. Following these summary statistics, the panel gives the results from the two non-parametric tests. [10]

The first non-parametric test measure corresponds to the sign test and gives the percentage of cases for which the absolute deviations of actual values from target values are lower than those of the classic M-vector model. The second non-parametric test measure corresponds to the Wilcoxon signed-rank test, which ranks the

[10] Focusing in negative deviations would result in an undesired observation reduction in the tests. Non-parametric tests of performance comparison using absolute deviations between effective and target returns (which are similar to our absolute deviations between effective and target terminal values – see footnote 9) have been also done by Bierwag et al. (1993) and Fooladi and Roberts (1992). Due to space constraints we report the tests only for the whole sample period.

differences between the absolute deviations from the generalized M-vector strategy and the classic M-vector strategy. Shown are the sum of the ranks corresponding to the cases where the absolute deviations from the generalized M-vector strategy are lower than the absolute deviations from the classic M-vector strategy (negative ranks), and the sum of ranks corresponding to the opposite cases (positive ranks). Asterisks accompanying the values of these two statistical measures of immunization performance indicate whether the specific generalized M-vector strategy performs different than (either better or worse) the classic M-vector strategy at 1% (**) or 5% (*) levels of statistical significance. [11]

Interesting patterns emerge as we inspect the results shown in Tables 2–6. First, the lower alpha generalized M-vector strategies outperform the classic M-vector strategy more frequently when the planning horizon is short. In fact, for the planning horizon of five years (see Table 6) and beyond, there is no evidence of improvement in immunization performance using the alternative M-vector strategies. [12] Later in this section, we provide some insights based on portfolio design that may partially explain the dependence of our results on the length of the planning horizon.

Table 2 reports the results corresponding to the shortest planning horizon of one year in our data set. Consistent with our previous results for the three-year horizon (in Table 1), we find no evidence of improvement in immunization performance when only a single M-vector constraint is used, once again confirming that the traditional duration is as good as any other single factor model. However, for all higher Qs there is consistent evidence of improvement (deterioration) in immunization performance at statistically significant levels for all generalized M-vector strategies with α less than one (with α more than 1). The improvement generally continues as α gets lower, and Q gets higher. However, it is important to point out that the significant improvement in immunization performance for low alpha strategies with $Q = 5$ may not be as relevant in an economic sense as it is in a statistical sense. This is because for $Q = 5$, the deviations are so low even for the classic M-vector model that further improvements may not be economically meaningful. However, for lower values of Q (especially for $Q = 2$ and 3 and possibly for $Q = 4$), gains in immunization performance are economically relevant. Of course, economic relevance can be assessed only in a relative sense, since what may be economically significant for a financial institution with hundreds of billions in assets and/or liabilities, may not be significant for institutions with a smaller monetary base.

As the length of the horizon increases we continue to find statistical evidence of improvement in immunization performance for generalized M-vector strategies with

[11] For example, a percentage of cases higher than 50% and marked by an asterisk indicates that the strategy performs statistically better than the classic M-vector strategy. On the other hand, a Sum of ranks $\pm$ marked by an asterisk and where the sum of negative ranks is higher than the sum of positive ranks points to the same conclusion.

[12] The results for $H = 6$ and 7 years can be obtained from the authors. These results show no evidence of improvement in immunization performance using alternative M-vector strategies, similar to reported in Table 6 for the case of $H = 5$.

Table 2
Immunization performance of the generalized M-vector model with $g(t) = t^{\alpha}$ over one-year horizons

Variable	$g(t) = t^{0.25}$	$g(t) = t^{0.5}$	$g(t) = t^{0.75}$	$g(t) = t$	$g(t) = t^{1.25}$	$g(t) = t^{1.5}$
Panel A: $Q = 1$						
Sum of absolute deviations	0.31680	0.25987	0.23972	0.26489	0.32648	0.39565
As percentage of $g(t) = t$	119.60	98.11	90.50	100.00	123.25	149.36
% of cases abs. dev. lower than $g(t) = t$	52.50	62.50	67.50*		20.00**	17.50**
Sum of ranks ±	371/449	475/345	547/273		130/690**	103/717**
Panel B: $Q = 2$						
Sum of absolute deviations	0.05216	0.05454	0.06786	0.09108	0.12504	0.17149
As percentage of $g(t) = t$	57.27	59.88	74.51	100.00	137.29	188.30
% of cases abs. dev. lower than $g(t) = t$	60.00	57.50	57.50		32.50*	25.00**
Sum of ranks ±	581/239*	591/229*	574/246*		180/640**	119/701**
Panel C: $Q = 3$						
Sum of absolute deviations	0.01687	0.02252	0.02687	0.03506	0.05384	0.08512
As percentage of $g(t) = t$	48.11	64.25	76.66	100.00	153.60	242.82
% of cases abs. dev. lower than $g(t) = t$	75.00**	67.50*	65.00		20.00**	12.50**
Sum of ranks ±	679/141**	623/197**	613/207**		95/725**	59/761**
Panel D: $Q = 4$						
Sum of absolute deviations	0.00287	0.00558	0.00986	0.01527	0.02090	0.02883
As percentage of $g(t) = t$	18.81	36.57	64.55	100.00	136.89	188.84
% of cases abs. dev. lower than $g(t) = t$	97.50**	92.50**	90.00**		22.50**	17.50**
Sum of ranks ±	816/4**	808/12**	800/20**		129/691**	144/676**
Panel E: $Q = 5$						
Sum of absolute deviations	0.00010	0.00026	0.00070	0.00179	0.00412	0.00844
As percentage of $g(t) = t$	5.59	14.27	39.39	100.00	230.46	471.84
% of cases abs. dev. lower than $g(t) = t$	95.00**	90.00**	87.50**		10.00**	7.50**
Sum of ranks ±	808/12**	788/32**	762/58**		33/787**	15/805**

This table reports different measures of the performance of 30 immunization strategies over 40 horizons of one year in the period 1950–1990. Each strategy is a result of combining a functional form of $g(t)$ with a generalized M-vector length (Q). Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of $g(t) = t$* indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks ±* indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

Table 3
Immunization performance of the generalized M-vector model with $g(t) = t^{\alpha}$ over two-year horizons

Variable	$g(t) = t^{0.25}$	$g(t) = t^{0.5}$	$g(t) = t^{0.75}$	$g(t) = t$	$g(t) = t^{1.25}$	$g(t) = t^{1.5}$
Panel A: $Q = 1$						
Sum of absolute deviations	0.35384	0.28471	0.25664	0.27921	0.34448	0.42578
As percentage of $g(t) = t$	126.73	101.97	91.92	100.00	123.37	152.49
% of cases abs. dev. lower than $g(t) = t$	46.15	56.41	66.67		17.95**	12.82**
Sum of ranks ±	330/450	425/355	501/279		127/653**	80/700**
Panel B: $Q = 2$						
Sum of absolute deviations	0.07234	0.06960	0.07929	0.09628	0.12552	0.16853
As percentage of $g(t) = t$	75.13	72.28	82.35	100.00	130.37	175.03
% of cases abs. dev. lower than $g(t) = t$	51.28	58.97	56.41		41.03	30.77*
Sum of ranks ±	455/325	516/264	523/257		206/574*	155/625**
Panel C: $Q = 3$						
Sum of absolute deviations	0.02688	0.03288	0.03746	0.04407	0.05874	0.08688
As percentage of $g(t) = t$	60.99	74.60	84.99	100.00	133.29	197.13
% of cases abs. dev. lower than $g(t) = t$	71.79*	64.10	66.67		28.21*	17.95**
Sum of ranks ±	570/210*	543/237*	543/237*		147/633**	82/698**
Panel D: $Q = 4$						
Sum of absolute deviations	0.00632	0.00895	0.01264	0.01657	0.02439	0.03666
As percentage of $g(t) = t$	38.14	53.99	76.28	100.00	147.19	221.21
% of cases abs. dev. lower than $g(t) = t$	71.79*	66.67	58.97		23.08**	15.38**
Sum of ranks ±	605/175**	577/203**	541/239*		116/664**	65/715**
Panel E: $Q = 5$						
Sum of absolute deviations	0.00044	0.00087	0.00166	0.00329	0.00566	0.00927
As percentage of $g(t) = t$	13.24	26.31	50.29	100.00	171.75	281.48
% of cases abs. dev. lower than $g(t) = t$	100.00**	94.87**	94.87**		28.21*	20.51**
Sum of ranks ±	780/0**	775/5**	761/19**		125/655**	99/681**

This table reports different measures of the performance of 30 immunization strategies over 39 horizons of two years in the period 1950–1990. Each strategy is a result of combining a functional form of $g(t)$ with a generalized M-vector length (Q). Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of* $g(t) = t$ indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks* ± indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

Table 4
Immunization performance of the generalized M-vector model with $g(t) = t^{\alpha}$ over three-year horizons

Variable	$g(t) = t^{0.25}$	$g(t) = t^{0.5}$	$g(t) = t^{0.75}$	$g(t) = t$	$g(t) = t^{1.25}$	$g(t) = t^{1.5}$
Panel A: $Q = 1$						
Sum of absolute deviations	0.42331	0.35757	0.33653	0.35701	0.39435	0.46796
As percentage of $g(t) = t$	118.57	100.16	94.26	100.00	110.46	131.08
% of cases abs. dev. lower than $g(t) = t$	52.63	65.79	65.79		31.58*	23.68**
Sum of ranks ±	373/368	438/303	456/285		252/489	176/565**
Panel B: $Q = 2$						
Sum of absolute deviations	0.08126	0.07856	0.09200	0.12320	0.15793	0.20358
As percentage of $g(t) = t$	65.96	63.77	74.68	100.00	128.19	165.24
% of cases abs. dev. lower than $g(t) = t$	73.68**	71.05*	76.32**		39.47	31.58*
Sum of ranks ±	578/163**	573/168**	588/153**		204/537*	178/563**
Panel C: $Q = 3$						
Sum of absolute deviations	0.04277	0.04168	0.04461	0.05019	0.06129	0.08751
As percentage of $g(t) = t$	85.22	83.04	88.88	100.00	122.12	174.36
% of cases abs. dev. lower than $g(t) = t$	55.26	57.89	57.89		36.84	26.32**
Sum of ranks ±	439/302	458/283	462/279		224/517*	141/600**
Panel D: $Q = 4$						
Sum of absolute deviations	0.02151	0.02473	0.02785	0.02966	0.03269	0.03797
As percentage of $g(t) = t$	72.52	83.38	93.90	100.00	110.22	128.02
% of cases abs. dev. lower than $g(t) = t$	68.42*	57.89	47.37		39.47	39.47
Sum of ranks ±	535/206*	470/271	397/344		322/419	273/468
Panel E: $Q = 5$						
Sum of absolute deviations	0.00258	0.00384	0.00572	0.00833	0.01106	0.01385
As percentage of $g(t) = t$	30.97	46.10	68.67	100.00	132.77	166.27
% of cases abs. dev. lower than $g(t) = t$	86.84**	84.21**	81.58**		34.21	26.32**
Sum of ranks ±	677/64**	666/75**	650/91**		197/544*	167/574**

This table reports different measures of the performance of 30 immunization strategies over 38 horizons of three years in the period 1950–1990. Each strategy is a result of combining a functional form of $g(t)$ with a generalized M-vector length (Q). Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of* $g(t) = t$ indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks* ± indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

Table 5
Immunization performance of the generalized M-vector model with $g(t) = t^{\alpha}$ over four-year horizons

Variable	$g(t) = t^{0.25}$	$g(t) = t^{0.5}$	$g(t) = t^{0.75}$	$g(t) = t$	$g(t) = t^{1.25}$	$g(t) = t^{1.5}$
Panel A: $Q = 1$						
Sum of absolute deviations	0.44019	0.37666	0.37532	0.41891	0.48903	0.57266
As percentage of $g(t) = t$	105.08	89.91	89.60	100.00	116.74	136.70
% of cases abs. dev. lower than $g(t) = t$	59.46	64.86	64.86		32.43*	27.03**
Sum of ranks ±	376/327	432/271	466/237		174/529**	142/561**
Panel B: $Q = 2$						
Sum of absolute deviations	0.09011	0.08502	0.09632	0.11834	0.15486	0.21176
As percentage of $g(t) = t$	76.14	71.85	81.39	100.00	130.86	178.95
% of cases abs. dev. lower than $g(t) = t$	62.16	62.16	59.46		40.54	27.03**
Sum of ranks ±	449/254	450/253	474/229		215/488*	142/561**
Panel C: $Q = 3$						
Sum of absolute deviations	0.03718	0.04463	0.05063	0.05583	0.06532	0.08959
As percentage of $g(t) = t$	66.61	79.94	90.69	100.00	117.01	160.47
% of cases abs. dev. lower than $g(t) = t$	56.76	64.86	56.76		45.95	35.14
Sum of ranks ±	465/238	453/250	422/281		250/453	186/517*
Panel D: $Q = 4$						
Sum of absolute deviations	0.02703	0.02390	0.02199	0.02221	0.02843	0.03640
As percentage of $g(t) = t$	121.66	107.58	99.01	100.00	127.97	163.88
% of cases abs. dev. lower than $g(t) = t$	45.95	37.84	51.35		37.84	29.73*
Sum of ranks ±	269/434	285/418	346/357		250/453	210/493*
Panel E: $Q = 5$						
Sum of absolute deviations	0.00777	0.00982	0.01202	0.01567	0.01735	0.01935
As percentage of $g(t) = t$	49.60	62.70	76.68	100.00	110.73	123.51
% of cases abs. dev. lower than $g(t) = t$	75.68**	75.68**	72.97**		48.65	37.84
Sum of ranks ±	614/89**	618/85**	551/152**		327/376	257/446

This table reports different measures of the performance of 30 immunization strategies over 37 horizons of four years in the period 1950–1990. Each strategy is a result of combining a functional form of $g(t)$ with a generalized M-vector length (Q). Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of* $g(t) = t$ indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks* ± indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

Table 6
Immunization performance of the generalized M-vector model with $g(t) = t^{\alpha}$ over five-year horizons

Variable	$g(t) = t^{0.25}$	$g(t) = t^{0.5}$	$g(t) = t^{0.75}$	$g(t) = t$	$g(t) = t^{1.25}$	$g(t) = t^{1.5}$
Panel A: $Q = 1$						
Sum of absolute deviations	0.45171	0.37355	0.38753	0.44478	0.53424	0.63071
As percentage of $g(t) = t$	101.56	83.98	87.13	100.00	120.11	141.80
% of cases abs. dev. lower than $g(t) = t$	55.56	61.11	63.89		27.78*	27.78*
Sum of ranks ±	370/296	431/235	442/224		147/519**	136/530**
Panel B: $Q = 2$						
Sum of absolute deviations	0.09296	0.10417	0.11310	0.12734	0.16357	0.22010
As percentage of $g(t) = t$	73.00	81.80	88.81	100.00	128.45	172.84
% of cases abs. dev. lower than $g(t) = t$	47.22	58.33	58.33		30.56*	30.56*
Sum of ranks ±	396/270	398/268	416/250		184/482*	145/521**
Panel C: $Q = 3$						
Sum of absolute deviations	0.04194	0.04828	0.05596	0.06111	0.07588	0.10289
As percentage of $g(t) = t$	68.63	79.00	91.57	100.00	124.18	168.37
% of cases abs. dev. lower than $g(t) = t$	61.11	66.67	58.33		41.67	33.33
Sum of ranks ±	429/237	434/232	392/274		201/465*	155/511**
Panel D: $Q = 4$						
Sum of absolute deviations	0.03021	0.02831	0.02651	0.02734	0.03610	0.04814
As percentage of $g(t) = t$	110.53	103.58	96.99	100.00	132.06	176.11
% of cases abs. dev. lower than $g(t) = t$	55.56	50.00	52.78		38.89	33.33
Sum of ranks ±	329/337	321/345	365/301		194/472*	176/490*
Panel E: $Q = 5$						
Sum of absolute deviations	0.01456	0.01574	0.01683	0.01736	0.01624	0.01795
As percentage of $g(t) = t$	83.88	90.67	96.95	100.00	93.52	103.38
% of cases abs. dev. lower than $g(t) = t$	66.67	63.89	52.78		61.11	47.22
Sum of ranks ±	449/217	414/252	367/299		395/271	298/368

This table reports different measures of the performance of 30 immunization strategies over 36 horizons of five years in the period 1950–1990. Each strategy is a result of combining a functional form of $g(t)$ with a generalized M-vector length (Q). Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of* $g(t) = t$ indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks* ± indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

α lower than 1, though the evidence becomes weaker as the horizon is lengthened. For a two-year horizon we find significant improvement in immunization performance for $Q = 3$, 4, and 5 (see Table 3); for a three-year horizon we find significant improvement in immunization performance for $Q = 2$, 4, and 5 (see Table 4); for a four-year horizon we find significant improvement in immunization performance for $Q = 5$ (see Table 5); and finally for a five-year horizon we do not find significant improvement in immunization performance for any Q (see Table 6) even though the magnitude of absolute deviations is lower for M-vector strategies with α lower than 1. [13]

Even for the cases when we find no statistical evidence of improvement in immunization performance, we find not a single instance when a generalized M-vector strategy with α lower than 1, performs significantly worse than the classic M-vector strategy. On the other hand, we find that all generalized M-vector strategies with α greater than 1 perform significantly worse than the classic M-vector strategy at all horizons, with only a few exceptions where statistical significance is not found (the exceptions occur at longer horizons and for higher values of Q).

Apart from these results, it should be noted that our choice of a polynomial function does not imply that this was the most optimal function for immunization. We also tested some other functions including Vasicek (1977) bond volatility function with different values of mean reversion and the natural log function. Except for the log function, other functions did not lead to immunization performance similar or better than that of the polynomial functions with low α. [14]

The results of immunization performance of the $g(t) = \log(t)$ model are shown in Table 7. The model performs very similar to the low α polynomial models, with significant gains in the immunization performance over short horizons and for values of Q greater than 2. The log function also satisfies the constraints on the first two derivatives of $g(t)$ with respect to the term to maturity stated at the end of Section 2. The results related to the log model suggest that there may be other functional forms of $g(t)$ that may lead to further improvements in immunization performance. However, we leave this task for future research in this area.

One unresolved question that still remains is why lower α strategies fail to outperform the classic M-vector strategy over longer planning horizons (i.e., $H = 5$ or more years). Though we do not have a convincing argument that explains this result, it should be noted that as the portfolios are rebalanced more often over a longer horizon, some of the immunization errors cancel out due to time diversification effects (positive errors are cancelled out with negative errors as the portfolio is rebalanced every year).

We investigate another possible explanation of the dependence of the immunization performance on the length of the planning horizon, based on a previous finding in the immunization literature on "portfolio design". Fig. 1 shows the average

[13] This remains true for even longer planning horizons of $H = 6$ and 7. These results can be obtained from the authors upon request.

[14] The results of immunization performance using the other functions can be obtained from the authors upon request. We are indebted to a referee for suggesting the log function.

Table 7
Immunization performance of the generalized M-vector model with $g(t) = \log(t)$

Variable	1-year horizons	2-year horizons	3-year horizons	4-year horizons	5-year horizons	6-year horizons
Panel A: $Q = 1$						
Sum of absolute deviations	0.39272	0.44267	0.52411	0.55351	0.57230	0.58189
As percentage of $g(t) = t$	148.26	158.54	146.80	132.13	128.67	130.30
% of cases abs. dev. lower than $g(t) = t$	47.50	41.03	44.74	54.05	52.78	45.71
Sum of ranks ±	308/512	257/523	303/438	318/385	311/355	283/347
Panel B: $Q = 2$						
Sum of absolute deviations	0.06133	0.08809	0.08754	0.11464	0.10831	0.13340
As percentage of $g(t) = t$	67.34	91.50	71.05	96.87	85.05	95.93
% of cases abs. dev. lower than $g(t) = t$	47.50	51.28	60.53	51.35	61.11	57.14
Sum of ranks ±	514/306	386/394	518/223*	392/311	389/277	341/289
Panel C: $Q = 3$						
Sum of absolute deviations	0.01087	0.02219	0.04381	0.03610	0.04037	0.05238
As percentage of $g(t) = t$	31.00	50.34	87.28	64.67	66.07	84.22
% of cases abs. dev. lower than $g(t) = t$	80.00**	71.79*	63.16	56.76	52.78	48.57
Sum of ranks ±	740/80**	617/163**	419/322	469/234	415/251	336/294
Panel D: $Q = 4$						
Sum of absolute deviations	0.00139	0.00443	0.01867	0.02915	0.02954	0.03762
As percentage of $g(t) = t$	9.09	26.72	62.94	131.20	108.07	102.19
% of cases abs. dev. lower than $g(t) = t$	100.00**	79.49**	73.68**	48.65	52.78	51.43
Sum of ranks ±	820/0**	643/137**	576/165**	300/403	344/322	299/331
Panel E: $Q = 5$						
Sum of absolute deviations	0.00004	0.00025	0.00202	0.00668	0.01230	0.01827
As percentage of $g(t) = t$	2.44	7.54	24.20	42.61	70.83	110.07
% of cases abs. dev. lower than $g(t) = t$	100.00**	100.00**	92.11**	81.08**	63.89	54.29
Sum of ranks ±	820/0**	780/0**	695/46**	632/71**	480/186*	348/282

This table reports different measures of the performance of the generalized M-vector model with $g(t) = \log(t)$ over horizons of different length in the period 1950–1990. Sum of ranks ± gives the sum of the ranks of the negative differences between the absolute deviations yielded by each strategy and those of the classic M-vector strategy and the sum of the ranks of the positive differences between both series. Asterisks accompanying the *% of cases each strategy absolute deviation is lower than that of* $g(t) = t$ indicate that the strategy performs statistically different from the classic M-vector strategy at 1% (**) or 5% (*) levels by means of a sign test on absolute deviations. Asterisks accompanying the *Sum of ranks* ± indicate that the strategy performs statistically different from the M-vector strategy at the preceding significance levels by means of a Wilcoxon test on absolute deviations.

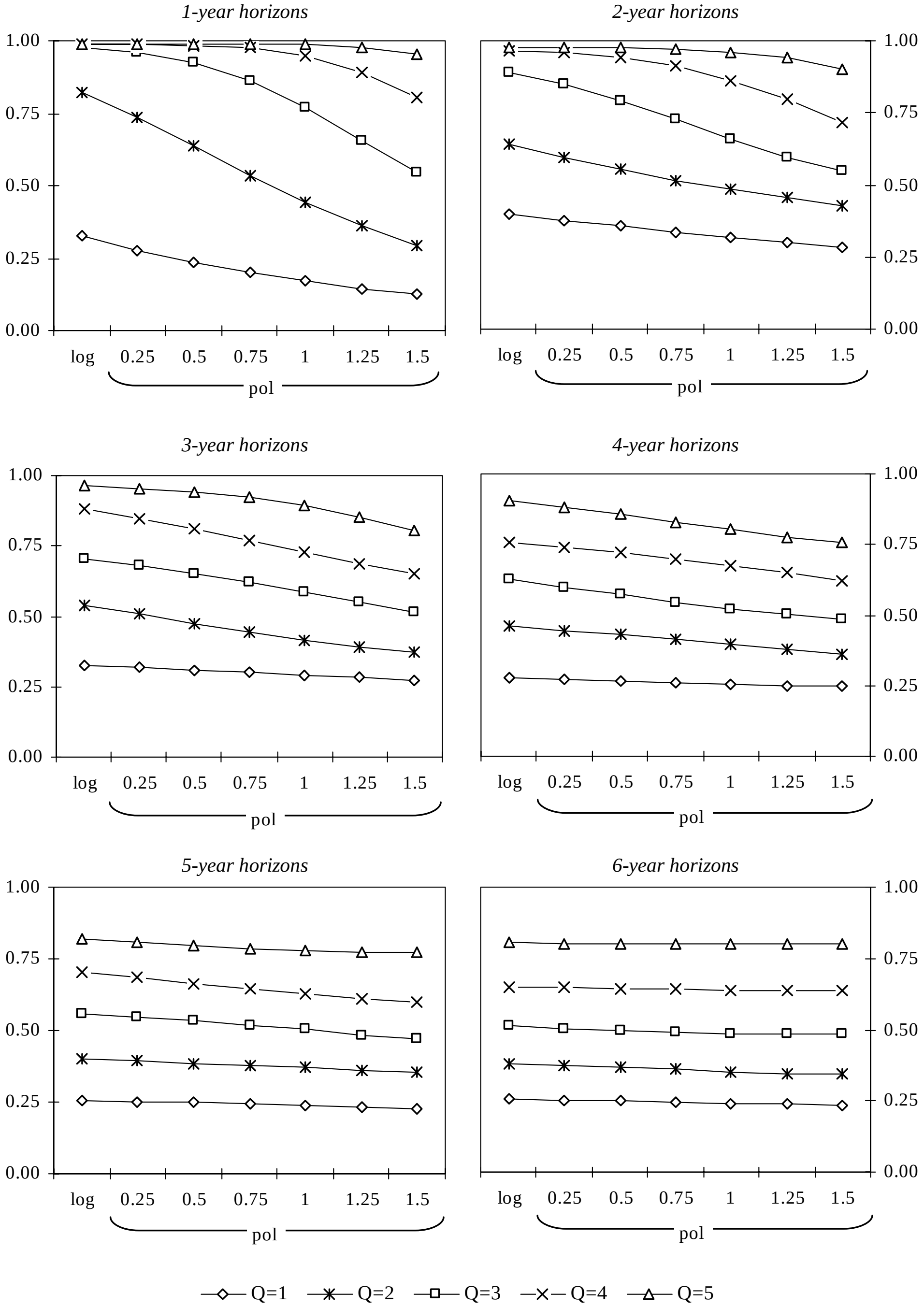

Fig. 1. Average investment in maturity bonds along the planning horizon.

investment in maturity bonds [15] (that is, the bonds which mature at the end of the horizon) along the planning horizon for different horizon lengths ranging from one to six years. In each of the six graphs, a specific line shows the investment in maturity bonds across models with different values of α for a given value of Q, and different lines denote different values of Q.

Inspection of these graphs reveals that lower α models select a higher proportion of bonds with maturities at the end of the horizon. However, this assertion weakens as the length of the horizon increases such that for long horizons of five and six years the differences between models are nearly insignificant. Coincidentally, the log model behaves very similar to the polynomial model with $\alpha = 0.25$, and so it allows us to plot it right next to this model. The pattern of immunization performance across models and horizon lengths is consistent with these graphs and corroborates the findings of Fooladi and Roberts (1992) and Bierwag et al. (1993) about the role of portfolio design in immunization performance.

4. Conclusions

This study is motivated by the need for improving interest rate hedging performance using finite length vectors of risk measures. While empirical studies have shown that the traditional duration vector models including the M-square model and the M-vector model offer good immunization performance, generalized M-vector models may lead to faster convergence of the true bond return function and therefore provide better immunization performance.

We have tested this hypothesis on generalized M-vector models corresponding to a class of polynomial functional forms given as $g(t) = t^{\alpha}$ where the value of α ranges from 0.25 to 1.5. Our empirical tests confirm that immunization results improve significantly for models $g(t) = t^{\alpha}$ with α between 0 and 1, when higher-order generalized M-vectors are used with short horizons (possibly due to infrequent portfolio rebalancing). Among other generalized M-vector models corresponding to different functional forms of $g(t)$, we found $\log(t)$ function to perform as well as the lower α polynomial models. The pattern of immunization performance across models and horizon lengths corroborates the findings of Fooladi and Roberts (1992) and Bierwag et al. (1993) about the role of maturity bonds in improving immunization performance.

The primary implication of our paper for bond portfolio management is that risk management may be improved by utilizing generalized M-vectors models. Applications would include short-term hedging, immunization over short planning horizons, and bond index replication.

[15] Since portfolios usually include short sales in bonds different from the maturity bonds, the investment in maturity bonds is defined as the portion of the sum of the squared bond weights (see Eq. (14)) which corresponds to bonds that mature at the end of the planning horizon.

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